TERM PAPER IN GEOMETRY
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EXERCISES 29, 30, 31, 71 (CHAPTER 9) IN M. J. GREENBERG'S "EUCLIDEAN AND NON-EUCLIDEAN GEOMETRIES"

INTRODUCTION

In 1892, Felix Klein made a important contribution to the study of geometry in his Erlanger programm. Klein unified and classified various geometrical studies of the nineteenth centrury by viewing geometry in general as the study of those properties of figures that remained invariant under the action of a particular group of tranformations on the underlying space (or manifold). So according to Klein the criterion that distinguishes one geometry from another is the group of transformations under which the propositions (of that geometry) remain true.

Klein provided several examples of geometries and their associated groups. Ordinary Euclidean geometry in two dimensions correspons to what Klein called "**principal group**". We might at first expect this to be the continuous group of all isometries (or congruent transformations). But since the propositions remain valid when the scale of measurement is altered, as in photographic enlargement, the "principal group" for Euclidean geometry (according to Klein) also includes "**similarities**" (which may change distances but preserve angles) in contrast to isometries which preserves length.

SIMILARITY IN THE EUCLIDEAN PLANE

A similarity is a transformation which takes each segment AB into segment A'B' whose lenght is given by A'B' = k * AB, where k is a constant positive number (the same for all segments) called the **ratio of magnification**. When k is equal to one the similarity is an isometry and when k is not equal to one the similarity is a dilation.

DILATION IN THE EUCLIDEAN PLANE

Let O be a point and k a number. The dilation with center O and ratio k is the transformation of the Euclidean plane that fixes O and maps a point P (not equal to O) onto the unique point P* on OP such that OP* = k(OP). In other words, points are moved radially from O a distance k times their original distance.

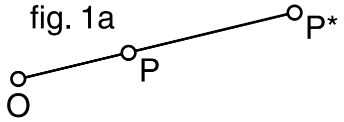


fig. 1a

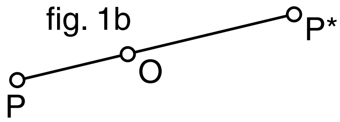


fig. 1b

The convention is that if k > 0 then P and P* are one the same side of the center O (fig 1a), and if k < 0 then P and P* are on the opposite sides of the center O (fig. 1b). Dilations will here be denoted D(O,k) with the center O and the ratio k.

REMARKS:

(i) A dilation is a transformation which preserves (or reverses) direction. It transforms each line into a parallel line.
(ii) A dilation is completely determined by its effect on any two given points.
(iii) k is a constant, independent of the position of O.
(iv) AB $\rightarrow$ AB is the identity.

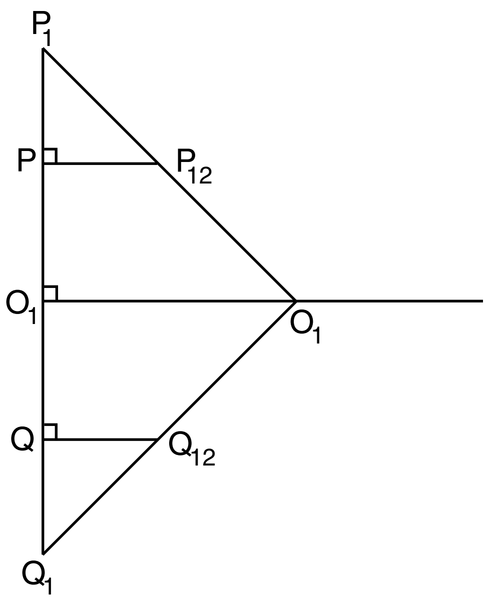
We can also view dilations with the help of Cartesian coordinates. This transformation is represented by (x,y) $\rightarrow$ (kx,ky).

EXERCISE 31, CHAPTER 9 - GROUP STRUCTURE OF DILATIONS AND TRANSLATIONS UNDER COMPOSITION

These results hold in Euclidean geometry. The algebraic properties of different operations in geometry are of great interest. It is a natural question to pose whether dilations form a group under composition.

A group <G,*> is a set G closed under the binary operation * such that:
G1. $\forall a,b,c \in G$, $(a*b)*c = a*(b*c)$ *associativity*
G2. $\exists e \in G$, $\forall a \in G$, $a*e = e*a = a$ *identity*
G3. $\forall a \in G$, $\exists a' \in G$, $a*a' = a'*a = e$ *inverse*

Obviously the existence of identity D(0,1) and inverse $D(O,k) \notin D(O,1/k)$ follows immediately. Dilations take points in the Euclidean plane and map them into the Euclidean plane. Thus associativity follows directly from the theorem of associativity of composition (c.f. any algebra textbook). G1, G2 and G3 hold. In order to constitute a group, the set of dilations has to be closed under composition.



EXAMPLE

Consider two distinct points P and Q. Draw the line between these two points. Find the midpoint 0_1. Let $D(O_1,k_1)$: $P \rightarrow P_1$, $D(0_1,k_1)$: $Q \rightarrow Q_1$. Construct the perpendicular to PQ at O_1. Let O_2 be a point on the perpendicular not being O_1. Let $D(0_2,k_2)$: $P_1 \rightarrow P_{12}$ with k_2 such that $PP_{12} \parallel O_1O_2$. Define Q_{12} by $D(0_2,k_2)$: $Q_1 \rightarrow Q_{12}$. $\Delta PP_1P_{12} \cong \Delta QQ_1Q_{12}$ implies $QQ_{12} \parallel PP_{12}$.

Assume that $D(0_2,k_2) \circ D(0_1,k_1) = D(0_3,k_3)$, but $QQ_{12} \parallel PP_{12}$ implies that there cannot be any such O_3. Thus the composition of two dilations does not necessarily has to be another dilation.

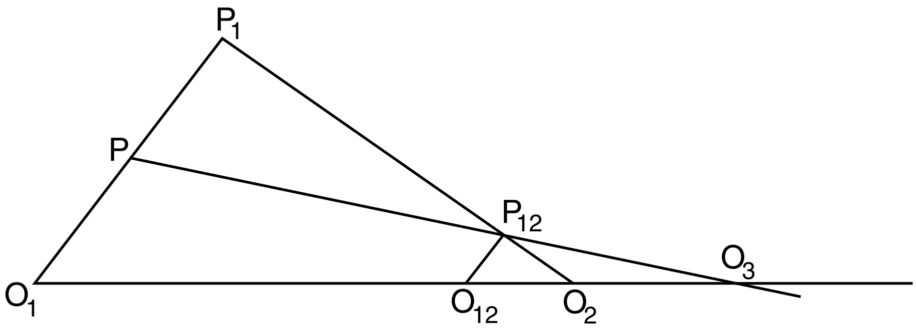
Considering this example, it is found to be a translation (take any point and dilate it twice as above, similarities of triangles give the result). It also shows that the composition $D(0_2,k_2) \circ D(0_1,k_1)$ is a translation if and only if $k_1k_2 = 1$. This raises a new question: Does the set of dilations and translations form a group under composition? Translations themselves form a commutative group under composition (proposition 9.12, Greenberg). The set of translations and dilations fulfills the requirements G1, G2 and G3. It remains to be proved that the set is closed under composition. Four different cases appear that have to be considered: $T_2 \circ T_1$, $D_2 \circ D_1$, $D_2 \circ T_1$, $T_2 \circ D_1$.

T_2 o T_1

$T_2 \circ T_1 = T_3$ since translations form a group under composition.

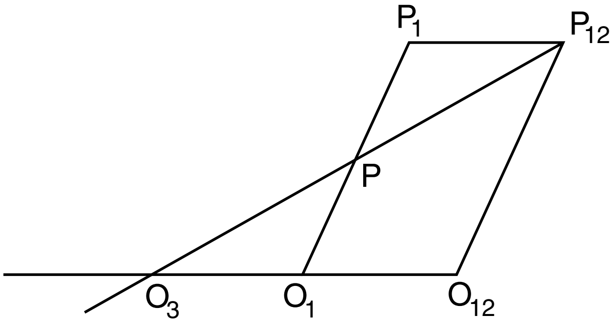
D_2 o D_1

Consider two arbitrary dilations $D_1(0_1,k_1)$, $D_2(0_2,k_2)$. $D_1(0_1,k_1)$: $O_1 \rightarrow O_1$.
The mapping of two distinct points defines a dilation completely. Consider an arbitrary point P. If $k_1k_2 = 1$, then $D_2 \circ D_1$ is a translation. Assume $k_1k_2 \neq 1$. If $D_1(0_1,k_1)$: $P \rightarrow P_1$, $D_2(0_2,k_2)$: $P_1 \rightarrow P_{12}$, $D_2(0_2,k_2)$: $O_1 \rightarrow O_{12}$, then the lines PP_{12} and O_1O_2 are not parallel. Hence these two lines have a point of intersection, say O_3. $D_2(0_2,k_2)$ is a dilation which means that $O_1P_1 \parallel O_{12}P_{12}$. $O_1P_1 = O_1P$, i.e. $O_1P \parallel O_{12}P_{12}$ and $\angle O_1OP = \angle O_{12}OP_{12}$. This means that there is a dilation $D_3(0_3,k_3) = D_2(0_2,k_2) \circ D_1(0_1,k_1)$. The composition of two dilations is a translation or a dilation.



T_2 o D_1

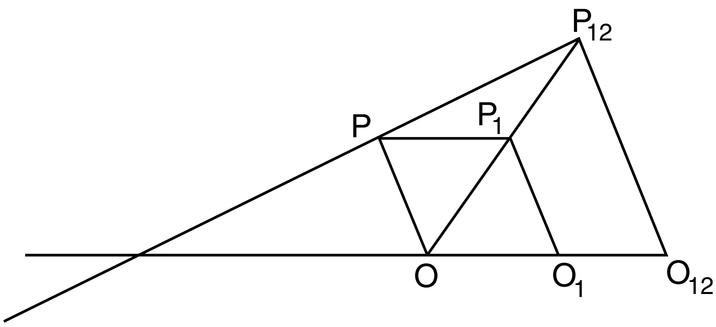
Consider an arbitrary point P. $D_1(0_1,k_1)$:$P \rightarrow P_1$, T_2: $O_1 \rightarrow O_{12}$, T_2: $P_1 \rightarrow P_{12}$. Assume $O_1O_{12} \parallel PP_{12}$. $P \in O_1P_1$ by D_1 being a dilation. $P \in P_1P_{12}$ by $O_1O_{12} \parallel P_1P_{12}$. Thus $P_1P_{12} \parallel PP_{12}$ and the lines have P_{12} in common which means that O_1, O_{12}, P, P_1 and P_{12} all lie on the same line. If $P = P_1$ then the composition is a translation. Otherwise it is a dilation with origin on the same line and ratio $t/(k_1-1)$ where t is the distance between a point and its translate. If O_1O_{12} is not parallel to PP_{12} then there exist a point O_3 belonging to both these lines.
$O_1P \parallel O_1P_1 \parallel O_{12}P_{12}$ by translation, $\angle O_1O_3P = \angle O_{12}O_3P_{12}$. The composition is a translation.



D_2 o T_1

Consider an arbitrary point P. T_1: $O \rightarrow O_1$, T_1: $P \rightarrow P_1$, $D_2(0,k_2)$: $P_1 \rightarrow P_{12}$, $D_2(0,k_2)$: $O_1 \rightarrow O_{12}$. Assume $OO_{12} \parallel PP_{12}$ and thus $PP_{12} \parallel OO_{12} = OO_1 \parallel PP_1$ and $PP_{12} = PP_1$. In the same way as in the previous case, this means that O, O_1, O_{12}, P, P_1 and P_{12} all belong to one line. D_2 being a dilation implies that $|O_{12}P_{12}| / |OO_{12}| = |O_1P_1| / |OO_1|$ which means that either $|O_{12}O| = |O_1O|$ and the composition is a translation or it is the special case $|O_1P_1| = |O_{12}P_{12}| = |OP| = O$, $O = P$ and then the composition is trivially a translation.

Assume OO_{12} is not parallel to PP_{12}. O_3 exists and so does a dilation $D_3(O_3, k_3)$ since $OP \parallel O_1P_1 \parallel O_{12}P_{12}$. In that case the composition is a dilation.



CONCLUSION

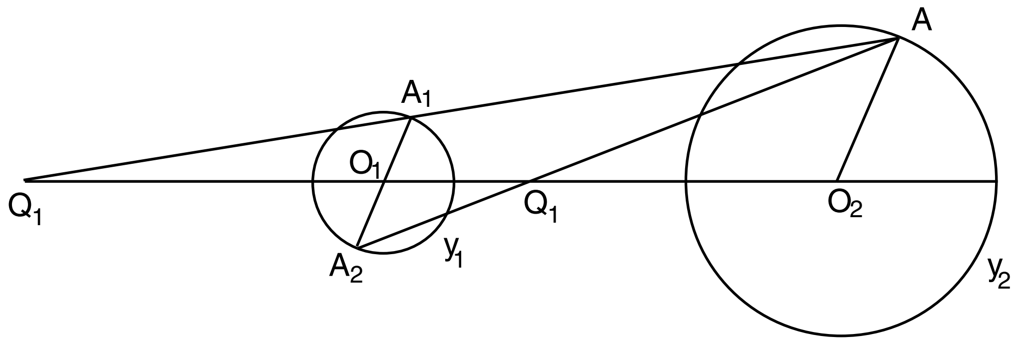
In all possible cases, the composition of translations and dilations is either a translation or a dilation. The set of these mappings is closed under composition, contains identity, inverses and the associative rule holds. Thus dilations and translations form a group under composition.

EXERCISE 29, CHAPTER 9 - EXISTENCE OF TWO DIFFERENT DILATIONS TAKING A CIRCLE ONTO ANOTHER

Consider two circles (y_1 and y_2) with two distinct radii and two distinct centers O_1, O_2. Choose an arbitrary point A on y_1, and construct the diameter A_1A_2 (with the points A_1, A_2 as the ends of the diameter) on y_2 such that A_1A_2 is parallel to O_2A (with A on the same side of O_1O_2 as A_1). Then the points Q_1 lies on A_1A and Q_2 on A_2A.

The triangle $Q_1O_1A_1$ is similar to the triangle Q_1O_2A and the triangle $A_2O_1Q_2$ is similar to the triangle Q_2AO_2. So the two circles are related by two dilations, D_1 (Q_1, O_2A / O_1A_1) and D_2 (Q_2, -O_2A / O_1A_1) whose centers Q_1 and Q_2 divide the segment O_1O_2 in the ratio $O_1A_1 : O_2A$.

The centers of Q_1 and Q_2 are called **centers of similitude** of the two circles.



EXERCISE 30, CHAPTER 9 -
A SPECIAL DILATION TAKING ∆ABC ONTO ∆A'B'C'

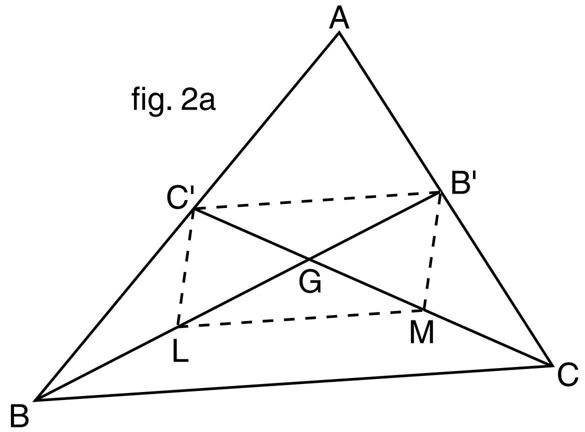
Consider two of the three medians, BB' and CC' and let them meet in the point G.

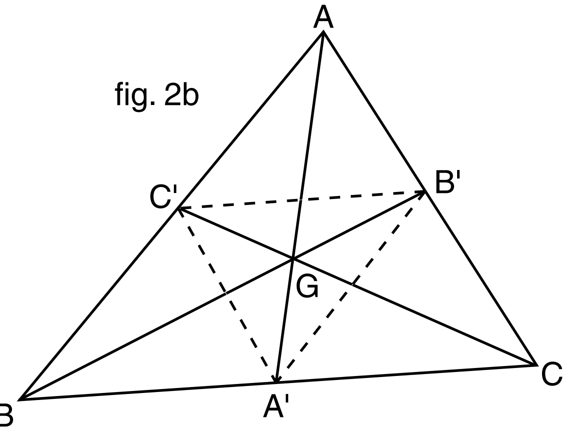
Let L and M be the midpoints of GB and GC. By Euclid VI.2 and Euclid VI.4 (see remarks), C'B' and LM are parallel to BC and C'B' and LM are half the length of BC.

The angles C'B'G and MLG are equal by Euclid I.29. So the triangle LC'B' are congruent with the triangle LMB' (by Euclid I.4). Thus LC' and MB' have the same lenght. It follows by Euclid I.27 that LC' and MB' are parallel.

So B'C'LM is a parallelogram (see figure 2a). Since the diagonals of a parallelogram bisect each other, we have B'G = GL = LB, C'G = GM = MC. Thus the two medians BB', CC' trisect each other at G. The point G is also a point of trisection of another, and similarly a third.

C'B' is half the length of BC, in the same way B'A' is half the length of BA and C'A' is half the length of CA. So the triangle A'B'C' is half the size of the triangle ABC. Thus the dilation (G, -1/2) takes the triangle A'B'C' onto the triangle ABC (see figure 2b).





REMARKS

- Euclid VI.2: *If a straight line is drawn parallel to one of the sides of a triangle, then it cuts the sides of the triangle proportionally; and, if the sides of the triangle are cut proportionally, then the line joining the points of section is parallel to the remaining side of the triangle* [http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI2.html].

- Euclid VI.4: *In equiangular triangles the sides about the equal angles are proportional where the corresponding sides are opposite the equal angles* [http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI4.html].

The common point G of the three medians is called the **centroid** of the triangle. Archimedes (287 - 212 B.C) obtained the centroid as the center of gravity of a triangular plate of uniform density in his work “On the Equilibrium of Plane Figures”.

EXERCISE 71, CHAPTER 9
A SPECIAL DILATION TAKING THE NINE-POINT CIRCLE ONTO THE CIRCUMSCRIBED CIRCLE

The nine-point circle can be constructed from any triangle ΔABC in the Euclidean plane. The medians AA', BB', CC' meet in the centroid G of ΔABC. D, E and F are the feet of the altitudes which meet in the ortocenter H. A', B', C', D, E, F and the midpoints of HA, HB and HC all lie on the nine-point circle.

From previous results it is known that the dilation D_1(G,-½) takes ΔABC onto ΔA'B'C'. This is equivalent to state the existence of another dilation D_2(G,-2) taking ΔA'B'C' onto ΔABC. The nine-point circle is by construction the circumscribed circle of the triangle ΔA'B'C'. In the Euclidean plane the circumscribed is uniquely determined by the triangle. The dilation D_2(G,-2) thus takes the nine-point circle onto the circumscribed circle.